On $K3$ surfaces which dominate Kummer surfaces

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Abstract

We study isogeny relation between complex algebraic $K3$ surfaces and Kummer surfaces by proving the following theorems: Kummer sandwich theorem for complex algebraic $K3$ surfaces with Shioda-Inose structure, and a Torelli-type theorem for the existence of rational maps from complex algebraic $K3$ surfaces to Kummer surfaces.

1 Introduction

In the present note we study rational maps between complex algebraic $K3$ surfaces in terms of their periods. Let X be a complex algebraic $K3$ surface and T_X be the transcendental lattice of X, which is endowed with a natural Hodge structure. For a natural number $n > 0$ let $T_X(n)$ be the lattice obtained by multiplying the quadratic form on T_X by n. One of fundamental problems for rational maps between $K3$ surfaces is the following question of Shafarevich, the origin of which goes back to the work of Inose and Shioda [7], [1].

Problem 1.1 ([5] Question 1.1)**.** *Let X and Y be complex algebraic $K3$ surfaces. Is it true that there exists a dominant rational map $X \dashrightarrow Y$ if and only if there exists a Hodge isometry $T_X \otimes \mathbb{Q} \simeq T_Y(n) \otimes \mathbb{Q}$ for some natural number n?*

Shafarevich's question is a kind of Torelli-type problem. Recall that the classical Torelli theorem for $K3$ surfaces asserts that there exists an isomorphism between two $K3$ surfaces if and only if their second integral cohomology lattices are Hodge isometric. In comparison with this Torelli theorem, Problem 1.1 proposes to consider the $\mathbb{Q}$-Hodge structures $T_X \otimes \mathbb{Q}$, modulo scaling of the quadratic forms, for the existence of rational maps between $K3$ surfaces. Problem 1.1 has been solved affirmatively for singular $K3$ surfaces, i.e., $K3$ surfaces with Picard number 20 by Inose and Shioda [7], [1], and for $K3$ surfaces with Picard number 19 by Nikulin-Shafarevich [5]. Nikulin [5] studied rational maps obtained as compositions of double coverings. The main purpose of this note is to answer Problem 1.1 affirmatively when the target Y is a Kummer surface.

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Theorem 1.2. *Let X and Y be complex algebraic $K3$ surfaces. Assume that Y is dominated by some Kummer surface, e.g., Y admits a Shioda-Inose structure or Y itself is a Kummer surface. Then there exists a dominant rational map $X \dashrightarrow Y$ if and only if there exists a Hodge isometry $T_X \otimes \mathbb{Q} \simeq T_Y(n) \otimes \mathbb{Q}$ for some natural number n.*

The proof of Theorem 1.2 will be based on three types of rational maps. The first two are (i) double coverings and (ii) rational maps between Kummer surfaces induced by isogenies of Abelian surfaces, which also have been used in [7], [1], and [5]. To prove Theorem 1.2, we will use furthermore (iii) relative multiplication maps between elliptic $K3$ surfaces and the associated basic elliptic surfaces.

Theorem 1.2 enables us to extend Inose's notion of isogeny [1], which was originally introduced for singular $K3$ surfaces, to those $K3$ surfaces which are dominated by Kummer surfaces. Obviously, Problem 1.1 is closely related to the notion of isogeny for $K3$ surfaces.

The original approach of Inose and Shioda for Problem 1.1 is to use the Kummer sandwich theorem for singular $K3$ surfaces, which roughly says that a singular $K3$ surface is isogenous to a Kummer surface in an explicit way. Recently Kummer sandwich theorem has been extended to a larger class of $K3$ surfaces ([6]) and has found some arithmetic applications. The $K3$ surfaces studied by Shioda [6] are characterized by the existence of Shioda-Inose structures such that the corresponding Abelian surfaces are products of elliptic curves. In this note we will prove Kummer sandwich theorem for all complex algebraic $K3$ surfaces with Shioda-Inose structure (Theorem 2.5). Therefore a $K3$ surface X with Shioda-Inose structure can be realized not only as a double covering of a Kummer surface Y but also as a double quotient of Y, which perhaps shed some light on isogeny relation between X and Y. However, as we rely on the Torelli theorem, our Kummer sandwich theorem is not explicit as in [1], [6], and our argument works only over $\mathbb{C}$.

Throughout this paper, the varieties are assumed to be complex algebraic. The transcendental lattice of an algebraic surface X will be denoted by T_X. By U we denote the rank 2 even indefinite unimodular lattice. By E_8 we denote the rank 8 even negative-definite unimodular lattice. For a lattice $L = (L, (,)_L)$ and a natural number n, we denote by $L(n)$ the scaled lattice $(L, n(,)_L)$.

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2 Kummer sandwich theorem

Let X be an algebraic $K3$ surface. Recall that a *Nikulin involution* of X is an involution $\iota : X \to X$ which acts trivially on $H^{2,0}(X)$. A Nikulin involution of X canonically corresponds to a double covering $X \dashrightarrow Y$ to another $K3$ surface. Indeed, if we have a double covering $\pi : X \dashrightarrow Y$, then the covering

transformation of π is a Nikulin involution of X. Conversely, for a Nikulin involution ι of X the minimal resolution $Y = \widetilde{X/\langle\iota\rangle}$ of the quotient surface is a $K3$ surface ([4]), and we have the rational quotient map $\pi : X \dashrightarrow Y$ of degree 2. The transcendental lattices T_X and T_Y are related by the chain of inclusions

$$2T_Y \subseteq \pi_* T_X = T_X(2) \subseteq T_Y, \tag{2.1}$$

which preserves the quadratic forms and the Hodge structures.

Nikulin [4], [5] and Morrison [3] developed the lattice-theoretic aspect of Nikulin involution. Let us denote

$$\Lambda_0 \ := \ E_8(2) \oplus U^3, \tag{2.2}$$

$$\Lambda_1 \ := \ \frac{1}{2}E_8(2) \oplus U^3. \tag{2.3}$$

We regard Λ_0 as a submodule of Λ_1 in a natural way. Then Λ_1 is the dual lattice of Λ_0. The following proposition reduces the construction of a Nikulin involution to a purely arithmetic problem.

Proposition 2.1 ([4], [3], [5])**.** *Let X be an algebraic $K3$ surface. Suppose that one is given a primitive embedding $T_X \subset \Lambda_0$ of lattices. Then there exists a Nikulin involution $\iota : X \to X$ such that, if we denote $Y = \widetilde{X/\langle\iota\rangle}$, then T_Y is Hodge isometric to the lattice*

$$T := (\, T_X \otimes \mathbb{Q} \, \cap \, \Lambda_1)\ (2), \tag{2.4}$$

where the Hodge structure of T is induced from T_X. Conversely, if one has a rational map $X \dashrightarrow Y$ of degree 2 to a $K3$ surface Y, then there exists a primitive embedding $T_X \subset \Lambda_0$ such that T_Y is Hodge isometric to the lattice T defined by (2.4).

Shioda-Inose structure is a particular kind of Nikulin involution.

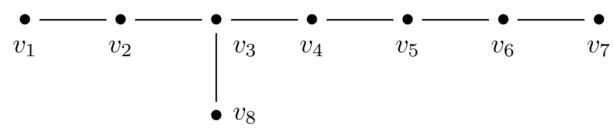
Definition 2.2 ([3])**.** An algebraic $K3$ surface X admits a *Shioda-Inose structure* if there exists a Kummer surface $Y = \mathrm{Km}A$ and a rational map $\pi : X \dashrightarrow Y$ of degree 2 such that π_* induces a Hodge isometry $T_X(2) \simeq T_Y$.

There is a lattice-theoretic characterization of $K3$ surfaces admitting Shioda-Inose structures due to Morrison.

Theorem 2.3 ([3])**.** *An algebraic $K3$ surface X admits a Shioda-Inose structure if and only if there exists a primitive embedding $T_X \hookrightarrow U^3$ of lattices.*

Shioda [6], extending the work of Inose [1], proved a Kummer sandwich theorem for elliptic $K3$ surfaces with sections and with two II^*-fibers. Shioda's construction uses the structure of elliptic fibration and thus works over an arbitrary algebraically closed field of characteristic $\neq 2, 3$. Over $\mathbb{C}$, the $K3$ surfaces studied by Shioda can be characterized by the existence of Shioda-Inose structures such that the corresponding Abelian surfaces are products of elliptic

curves. Here we shall derive, in a transcendental way, a Kummer sandwich theorem for all complex algebraic $K3$ surfaces with Shioda-Inose structure. Denote the Dynkin diagram of E_8 by



We identify the $\mathbb{Z}$-modules underlying E_8 and $E_8(2)$ in a natural way, and regard the above set $\{v_i\}_{i=1}^8$ as a basis of $E_8(2)$. Then we have $(v_i, v_i) = -4$ for $i = 1, \cdots, 8$, $(v_i, v_j) = 2$ if v_i and v_j are connected by a segment, and $(v_i, v_j) = 0$ otherwise. Let $\{e_i, f_i\}_{i=1}^3$ be the standard basis of U^3. We have $(e_i, e_j) = (f_i, f_j) = 0$ and $(e_i, f_j) = \delta_{ij}$. For $i = 1, 2, 3$, we define the vectors $l_i, m_i \in \Lambda_0 = E_8(2) \oplus U^3$ by

$$\begin{aligned} l_1 &= -v_5 + v_7 + 2(e_1 + f_1), \\ m_1 &= -v_4, \\ l_2 &= v_1 + v_8 + 2(e_2 + f_2), \\ m_2 &= v_2, \\ l_3 &= v_7 + v_8 + 2(e_1 + e_2 + e_3 + f_3), \\ m_3 &= v_6, \end{aligned}$$

and put $L := \langle l_1, m_1, l_2, m_2, l_3, m_3 \rangle$.

Lemma 2.4. *The sublattice $L \subset \Lambda_0$ has the following properties.*

(1) $L \simeq U(2)^3$.
(2) $L \subset 2\Lambda_1$.
(3) *L is a primitive sublattice of Λ_0.*

Proof. We can extend the set $\{l_i, m_i\}_{i=1}^3$ to a basis of Λ_0 by adding the set of vectors $\{v_3, v_5, e_1, f_1, e_2, f_2, e_3, f_3\}$. Thus L is primitive in Λ_0. The assertion (2) is obvious and the assertion (1) is proved by direct calculations. □

Theorem 2.5. *Let X be an algebraic $K3$ surface admitting a Shioda-Inose structure $X \dashrightarrow Y = \mathrm{Km}A$. Then there exists a Nikulin involution ι on Y such that the minimal resolution of the quotient surface $Y/\langle\iota\rangle$ is isomorphic to X. In particular, one has the following sequence of rational maps of degree 2:*

$$\mathrm{Km}A \dashrightarrow X \dashrightarrow \mathrm{Km}A.$$

Proof. By the definitions, we have the Hodge isometries

$$T_Y \simeq T_A(2), \quad T_X \simeq T_A.$$

Since T_A is embedded into $H^2(A, \mathbb{Z}) \simeq U^3$ primitively, there exists a primitive embedding $\varphi : T_Y \hookrightarrow U(2)^3$. By composing φ with an isometry $U(2)^3 \simeq L$, we obtain a primitive embedding $\psi : T_Y \hookrightarrow \Lambda_0$ such that $\psi(T_Y) \subset 2\Lambda_1$. We have

$$\psi(T_Y) \otimes \mathbb{Q} \, \cap \, \Lambda_1 \; = \; \frac{1}{2}\psi(T_Y).$$

By Proposition 2.1, there exists a Nikulin involution $\iota : Y \to Y$ such that for the minimal resolution Z of $Y/\langle\iota\rangle$ the transcendental lattice T_Z is Hodge isometric to

$$\frac{1}{2}T_Y(2) \simeq \frac{1}{2}T_A(4) \simeq T_A \simeq T_X.$$

Since a Hodge isometry $T_Z \simeq T_X$ can be extended to a Hodge isometry $H^2(Z,\mathbb{Z}) \simeq H^2(X,\mathbb{Z})$ (cf. [3]), we have $Z \simeq X$ by the Torelli theorem. $\square$

The rational quotient map $\pi : \mathrm{Km}A \dashrightarrow X$ constructed in Theorem 2.5 induces a Hodge isometry $\pi^* : T_X(2) \to T_{\mathrm{Km}A}$. Thus a $K3$ surface X with Shioda-Inose structure can be defined not only as a particular covering of a Kummer surface $\mathrm{Km}(A)$ but also as a particular quotient of $\mathrm{Km}(A)$. In particular, X is dominated by A by a rational map of degree 4.

Now, is it true in general that a double covering X of a Kummer surface $\mathrm{Km}A$ admits a double covering $\mathrm{Km}A \dashrightarrow X$ of the opposite direction, as like the situation for elliptic curves?

Example 2.6. Let A be an Abelian surface with $T_A \simeq U \oplus \langle 2\rangle \oplus \langle -2\rangle$ and X be the $K3$ surface with T_X Hodge isometric to $2T_A$. Then there exists a rational map $X \dashrightarrow \mathrm{Km}A$ of degree 2, but there does *not* exist rational map $\mathrm{Km}A \dashrightarrow X$ of degree 2.

Proof. We first construct a double covering $X \dashrightarrow \mathrm{Km}A$. Define the vectors $w_1, \cdots, w_4 \in \Lambda_0$ by

$$\begin{aligned}
w_1 &:= v_3 + v_7 + 2(e_1 - f_1 + 2e_2 + f_2),\\
w_2 &:= -v_2 - v_6 + 2(f_2 - e_3 - f_3),\\
w_3 &:= v_1 - v_3 + 2(2e_1 + f_1),\\
w_4 &:= v_4 + v_8 + 2e_1.
\end{aligned}$$

Similarly as Lemma 2.4, we will see that the vectors $w_1, \cdots, w_4$ define a primitive embedding $\varphi : T_X \hookrightarrow \Lambda_0$ such that

$$\varphi(T_X) \otimes \mathbb{Q} \cap \Lambda_1 = \frac{1}{2}\varphi(T_X).$$

By Proposition 2.1 and the Torelli theorem, we obtain a double covering $X \dashrightarrow \mathrm{Km}A$.

Assume that we have a rational map $\mathrm{Km}A \dashrightarrow X$ of degree 2. By Proposition 2.1, there exists a primitive embedding $T_A(2) \hookrightarrow \Lambda_0$ such that

$$T_A(2) \otimes \mathbb{Q} \cap \Lambda_1 = T_A(2). \tag{2.5}$$

Via this embedding, we regard $T_A(2)$ as a primitive sublattice of Λ_0. Let $\pi : T_A(2) \to U^3$ be the orthogonal projection, which is injective by the condition (2.5). Let M be the lattice $\pi(T_A(2))$ and N be the primitive closure of M in U^3. By the condition (2.5) again, the Abelian group N/M has no 2-component. For an even lattice L let $L^\vee$ be the dual lattice of L, $D_L = L^\vee/L$ be the discriminant

group of L, and $(D_L)_2$ be the 2-component of D_L. We see from the inclusions $M \subset N \subset N^\vee \subset M^\vee$ that

$$(D_M)_2 \simeq (D_N)_2 \simeq (D_{N^\perp \cap U^3})_2. \tag{2.6}$$

The second isomorphism follows from the fact that N is a primitive sublattice of the unimodular lattice U^3. In particular, the length of $(D_M)_2$ is less than or equal to 2. On the other hand, we have $(v, w) \in 2\mathbb{Z}$ for every $v, w \in M$. Thus we have $\frac{1}{2}M \subset M^\vee$, which is absurd. □

Remark 2.7. It follows from [5] Theorem 1.3 that for X and A as in Example 2.6, there nevertheless exists a rational map $\mathrm{Km}A \dashrightarrow X$ of degree 2^μ for some $\mu > 1$.

3 Rational maps to Kummer surfaces

In this section we study rational maps from $K3$ surfaces to Kummer surfaces in general. We "rationalize" Morrison's criterion (Theorem 2.3) as follows.

Proposition 3.1. *For an algebraic $K3$ surface X the following conditions are equivalent.*

(i) *There exists a dominant rational map $X \dashrightarrow \mathrm{Km}A$ to some Kummer surface* $\mathrm{Km}A$.

(ii) *There exists an embedding $T_X \otimes \mathbb{Q} \hookrightarrow U^3 \otimes \mathbb{Q}$ of quadratic spaces.*

(iii) *There exists an embedding $T_X \hookrightarrow U^3$ of lattices.*

Proof. (i) ⇒ (ii): A rational map $f : X \dashrightarrow \mathrm{Km}A$ of finite degree d induces a Hodge isometry

$$f_* : T_X(d) \otimes \mathbb{Q} \overset{\simeq}{\to} T_{\mathrm{Km}A} \otimes \mathbb{Q} \simeq T_A(2) \otimes \mathbb{Q}.$$

Thus $T_X \otimes \mathbb{Q}$ is isometric to $T_A(2d) \otimes \mathbb{Q}$. Since $T_A(2d) \otimes \mathbb{Q}$ is embedded into $H^2(A, \mathbb{Q})(2d) \simeq U^3(2d) \otimes \mathbb{Q} \simeq U^3 \otimes \mathbb{Q}$, we obtain an embedding $T_X \otimes \mathbb{Q} \hookrightarrow U^3 \otimes \mathbb{Q}$ of quadratic spaces.

(ii) ⇒ (iii): When $\mathrm{rk}(T_X) \leq 3$, the lattice T_X can always be embedded (primitively) into U^3. Thus we may assume that $\mathrm{rk}(T_X) = 4$ or 5. When $\mathrm{rk}(T_X) = 4$, the condition (ii) is equivalent to the existence of an isotropic vector in T_X. Let T be a maximal even overlattice of T_X. Since T also contains an isotropic vector, we can write $T \simeq U \oplus L$ for some rank 2 lattice L. Then T can be embedded primitively into U^3. When $\mathrm{rk}(T_X) = 5$, the condition (ii) is equivalent to the existence of a rank 2 totally isotropic sublattice of T_X. Similarly as the case of $\mathrm{rk}(T_X) = 4$, every maximal even overlattice of T_X is of the form $T = U^2 \oplus L$, $\mathrm{rk}(L) = 1$, and thus can be embedded primitively into U^3.

(iii) ⇒ (i): We fix an embedding $T_X \subset U^3$. Let T be the primitive closure of T_X in U^3 and endow T with the Hodge structure induced from T_X. We regard T as a primitive sublattice of $U^3 \oplus E_8^2$. By the surjectivity of the period map, there

exists a $K3$ surface Y with T_Y Hodge isometric to T. We have an embedding $T_X \hookrightarrow T_Y$ of finite index which preserves the periods. The Néron-Severi lattice NS_X of X admits an embedding of U because we have $\mathrm{rk}(NS_X) \geq 13$. It follows from [2] Proposition 7 that there exists a dominant rational map $X \dashrightarrow Y$. Since the lattice T_Y can be embedded primitively into U^3, the $K3$ surface Y admits a Shioda-Inose structure $Y \dashrightarrow \mathrm{Km}A$ by Theorem 2.3. In this way we constructed a rational map $X \dashrightarrow \mathrm{Km}A$ of finite degree. $\square$

An Abelian surface A is a product of two elliptic curves if and only if T_A can be embedded primitively into U^2. Thus we have the following variant of Proposition 3.1.

Proposition 3.2. *For an algebraic $K3$ surface X the following conditions are equivalent.*

(i) *There exists a dominant rational map $X \dashrightarrow \mathrm{Km}A$ to some Kummer surface $\mathrm{Km}A$, where A is a product of two elliptic curves.*

(ii) *There exists an embedding $T_X \otimes \mathbb{Q} \hookrightarrow U^2 \otimes \mathbb{Q}$ of quadratic spaces.*

(iii) *There exists an embedding $T_X \hookrightarrow U^2$ of lattices.*

By using Proposition 3.1 we deduce the next theorem, from which Theorem 1.2 follows immediately.

Theorem 3.3. *Let X be an algebraic $K3$ surface and $\mathrm{Km}A$ be an algebraic Kummer surface. Then there exists a dominant rational map $X \dashrightarrow \mathrm{Km}A$ if and only if there exists a Hodge isometry $T_X \otimes \mathbb{Q} \simeq T_A(n) \otimes \mathbb{Q}$ for some natural number n.*

Proof. It suffices to prove the "if" part. Assume the existence of a Hodge isometry $T_X \otimes \mathbb{Q} \simeq T_A(n) \otimes \mathbb{Q}$. As $T_A \otimes \mathbb{Q}$ is embedded into $U^3 \otimes \mathbb{Q}$, by Proposition 3.1 we can find a Kummer surface $\mathrm{Km}B$ and a finite rational map $X \dashrightarrow \mathrm{Km}B$. Since we have a Hodge isometry $T_B(m) \otimes \mathbb{Q} \simeq T_A \otimes \mathbb{Q}$ for some natural number m, the Abelian surface B is isogenous to the Abelian surface A. Thus there exists a dominant rational map $\mathrm{Km}B \dashrightarrow \mathrm{Km}A$. $\square$

Corollary 3.4. *Let X be an algebraic $K3$ surface and A be an Abelian surface. If we have a dominant rational map $A \dashrightarrow X$, then there exists a dominant rational map $X \dashrightarrow \mathrm{Km}A$.*

Note that there does not exist non-constant rational map $X \dashrightarrow A$ because we have $h^1(\mathcal{O}_X) = 0$. The Kummer sandwich diagram (2.5) is an example of this Corollary 3.4.

Corollary 3.5. *Let X and Y be algebraic $K3$ surfaces dominated by some Kummer surfaces. Then there exists a dominant rational map $X \dashrightarrow Y$ if and only if there exists a dominant rational map $Y \dashrightarrow X$ of the opposite direction.*

Thus, as like Inose's paper [1], we are able to define a notion of isogeny for $K3$ surfaces dominated by Kummer surfaces by the existence of dominant rational map.

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